Cases of Hyperdimensional Awareness

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H. S. M. Coxeter

Of great interest to the investigation of creativity and nusual states of consciousness are the available records of persons who ere able not only to conceive and imagine higher dimensional forms, but o do so exactly enough as to make new discoveries which then could be ested by logical demonstration even by persons unendowed with such ifts. An important noëtic distinction between discovery and proof arises, ere well illustrated. Discovery involves more creative power than proof, ust as proof requires more than explanation; and explanation, more than ranscription or repetition. Sometimes all these levels are present in one erson.

Every student of these matters must be indebted to Dr. H.S.M. Coxeter Ph.D.) for having collected between the two covers of one book (Regular 'olytopes, *Methuen, London, 1948, now in a second edition by Macmillan, 1963) so many remarkable examples of unusually powerful tates of consciousness, resulting in new, strange, yet always verifiable orms. The following biographical sketches are collected here. "Polytopes" neans closed geometrical figures, usually solids of more than three limensions.*

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Alicia Boole Stott

Alicia Boole Stott (1860-1940) was the middle one of George Boole's five daughters. Her father, who is famous for his algebra

of logic and his textbook on Finite Differences, died when she was four years old; so her mathematical ability was purely hereditary. She spent her early years, repressed and unhappy, with her maternal grandmother and great-uncle in Cork.

When Alice was about thirteen, the five girls were reunited with their mother (whose books reveal her as one of the pioneers of modern pedagogy) in a poor, dark, dirty, and uncomfortable lodging in London. There was no possibility of education in the ordinary sense, but Mrs. Boole's friendship with James Hinton attracted to the house a continual stream of social crusaders and cranks.

It was during those years that Hinton's son Howard brought a lot of small wooden cubes, and set the youngest three girls the task of memorizing the arbitrary list of Latin words by which he named the cubes, and piling them into shapes. To Ethel, and possibly Lucy too, this was a meaningless bore; but it inspired Alice (at the age of about eighteen) to an extraordinarily intimate grasp of four-dimensional geometry . . . . Her methods remained purely synthetic, for the simple reason that she had never learnt analytical geometry.

In 1890 she married Walter Stott, an actuary; and for some years she led a life of drudgery, rearing her two children on a very small income. Meanwhile, in Holland, P. Schoute was describing the central sections of the six regular four-dimensional polytopes.* Mr. Stott drew his wife's attention to Schoute's published work; so she wrote to say that she had already determined the whole sequence of sections, the middle section for each polytope agreeing with Schoute's result.

In an enthusiastic reply, he asked when he might come over to England and work with her. He arranged for the publication of her discoveries in 1900, and a friendly collaboration continued

*The sixth—the one without a parallel among the five regular three-dimensional solids—is the most unusual of all the regular hyper-solids in four dimensions. It is the higher analogue of the three-dimensional cuboctahedron—which is the semi-regular solid whose twelve vertices lie at the centers of twelve equal spheres all packed around a thirteenth. Just as only six equal coins fit around a seventh in their midst, their six centers forming the vertices of a hexagon, so just twelve ping pong balls will fit around a thirteenth. In four dimensions, twenty-four hyperspheres will just fit around a twenty-fifth, the centers of the twenty-four forming the vertices of the polytope in question. It has also twenty-four octahedral cells. *C.M.*

for the rest of his life. Her cousin, Ethel Everest, used to invite them to her house at Hever, Kent, where they spent many happy summer holidays. Mrs. Stott's power of geometrical visualization supplemented Schoute's more orthodox methods, so they were an ideal team. After his death in 1913 she attended the tercentenary celebrations of his university of Groningen, which conferred upon her an honorary degree, and exhibited her models.

She resumed her mathematical activities in 1930, when her nephew, the physicist G. I. Taylor, introduced her to me. The strength and simplicity of her character combined with the diversity of her interests to make her an inspiring friend. She collaborated with me in the investigation of Gosset's semi-regular four-dimensional polytope, which I had rediscovered about that time. She made models of its sections, which are probably still on view in Cambridge.

Paul S. Donchian

Paul S. Donchian was born in America of Armenian parentage. His great-grandfather was a jeweller at the court of the Sultan of Turkey, and many of his other ancestors were oriental jewellers and handicraftsmen. He was born in Hartford, Connecticut, in 1895. His mathematical training ended with high school geometry and algebra, but he was always interested in scientific subjects. He inherited the rug business established by his father, and began operating it in 1920.

At about the age of thirty he suddenly began to experience a number of startling and challenging dreams of the previsionary type soon afterward to be described by J. W. Dunne in "An Experiment With Time." In an attempt to solve the problems thus presented, he determined to make a thorough analysis of the geometry of hyper-space.

His aim was to reduce the subject to its simplest terms, so that anyone like himself with only elementary mathematical training could follow every step. For this purpose he devoted many years to the task of making a set of exquisite models. Their

construction required all the patience and delicate craftsma ship that could be provided by his oriental background. In th complicated cases it was not feasible to superpose section because frequently the edges of a section are not edges of th polytope. He took the central section as an "exterior shell," b for the rest he made use of various plane projections publishe by Schoute and van Oss, regarding them like an architect as pl and elevation.

He observed that any solid projection may be regarded as a intermediate stage in the formation of a plane projection, whic means that the solid projection should present the appearanc of a plane projection when viewed from far away in a direction. To quote Donachian's own words:

> My system is to build first the central grouping, then the exter shell, with the central grouping inserted at the last moment a suspended by temporary stay-cords. The process of connecting t innermost and outermost portions proceeds by constant testing the results [by comparison with the known plane projections] a the plodding application of common sense.• The models a fortunately fool-proof, because if a mistake is made it is immedia apparent and further work is impossible. The final joining of t inner and outer portions carries something of the thrill experienc by two tunnelling parties, piercing a mountain from opposite sid when they finally break through and find that their diggings a exactly in line.

In 1934 the models were exhibited at the Century of Progre Exposition in Chicago and at the Annual Exhibit of th American Association for the Advancement of Science, Pittsburgh.

Thorold Gosset

Thorold Gosset was born in 1869. After a largely classic schooling, he went up to Pembroke College, Cambridge, 1888. He was called to the Bar in 1895, and took a law degr the following year. Then, having no clients, he amused hims

by trying to find out what regular figures might exist in n dimensions.

After rediscovering all of them, he proceeded to enumerate the "semi-regular" figures. He recorded the results in an essay, which he sent to J. Glaisher in 1897. Glaisher showed it to Alfred North Whitehead and William Burnside. It is tempting to speculate on the possibility that some of its ideas, unconsciously assimilated, bore fruit in Burnside's later work. This, however, is unlikely; for Burnside declared (in a letter to Glaisher, dated 1899) that he never found time to read more than the first half. [It is also a pity that Whitehead, co-author of the *Principia Mathematica* with Bertrand Russell, did not appreciate Gosset's profound mathematical insights.]

"The author's method, a sort of geometrical intuition," did not appeal to Burnside, and the idea of regarding an (n-l)-dimensional lattice as a degenerate n-dimensional polytope seemed "fanciful." He thus failed to appreciate the new discoveries, and Glaisher was content to publish the barest outline. That published statement remained unnoticed until after its results had been rediscovered by Elte and myself. As he was a modest man, Gosset let the subject drop, and pursued his career as a lawyer. [Gosset was also the first to conceive of a· measure semi-regular shapes in higher dimensions.]

John Flinders Petrie

John Flinders Petrie, who first realized the importance of skew polygon that now bears his name, is the only son of Sir W. M. Flinders Petrie, the great Egyptologist. He was born in 1907, and as a schoolboy showed remarkable promise of mathematical ability.

In periods of intense concentration he could answer questions about complicated four-dimensional figures by "visualizing" them. His skill as a draftsman can be seen in his unique set of drawings of stellated icosahedra.

In 1926, he generalized the concept of a regular skew polygon to that of a regular skew polyhedron. The theory is complete

only up to four dimensions, and the analogous "skew polytopes" have not been investigated at all.

Ludwig Schläfli

Practically all the ideas in this chapter (with the exception of Schoute's generalized prism or rectangular product) are due to Schläfli, who discovered them before 1853—a time when A. Cayley, H. Grassmann, and A. Möbius were the only other people who had ever conceived the possibility of geometry in more than three dimensions.* [John Graves, who had first conceived Cayley's later eight-dimensional algebra by December 1843, should also be mentioned, as well as Graves' friend W.R. Hamilton, discoverer of four-dimensional quaternion algebra.]

Ludwig Schläfli was born in Grasswyl, Switzerland, in 1814. In his youth he studied science and theology at Berne, but received no adequate instruction in mathematics. From 1837 till 1847 he taught in a school at Thun, and learned mathematics in his spare time, working quite alone until his famous compatriot, the geometer Jakob Steiner, introduced him to Jacobi and Dirichlet. Then he was appointed a lecturer in mathematics at the University *(Hochschule)* of Berne, where he remained for the rest of his long life.

His pioneering work was so little appreciated in his time that only two fragments of it were accepted for publication: one in France and one in England.

The French and English abstracts of this work, which were published in 1855 and 1858, attracted no attention. This may have been because their dry-sounding titles tended to hide the geometrical treasures that they contain, or perhaps it was just because they were ahead of their time, like the art of van Gogh. Anyhow, it was nearly thirty years later that some of the same ideas were rediscovered by an American. The latter treatment (W. I. Stringham, 1880) was far more elementary and perspicuous, being enlivened by photographs of models and by

*Möbius realized, as early as 1827, that a four-dimensional rotation would be required to bring two enantiomorphous [e.g. a right and a left-hand glove] solids into coincidence. This idea was neatly employed by H. G. Wells in *The Plattner Story*.

drawings . . . .The result was that many people imagined Stringham to be the original discoverer of the regular polytopes. [Schläfli had not only been the first to conceive of regular hyperdimensional forms, but was also the first man in recorded history to compute and measure the content and hypersurface of a higher dimensional sphere. Ways of closest packings of hyperspheres, though adumbrated by Gosset, were not explicitly investigated until Minkowski in 1904. Not until 1963 was it begun to be clearly realized that packing patterns changed remarkably after eight dimensions, and Musès' concise single formula discovered then for the packing of spheres up to and including 8-dimensional space will be found in Coxeter's chapter in Vol. III of *Lectures in Modern Mathematics* edited by T. L. Saaty, 1965.]

As evidence that at last the time was ripe, we may mention the independent rediscovery [of regular higher dimensional solids] between 1880 and 1900, by G. Forchhammer, K. Rudel, R. Hoppe, V. Schlegel, A. Puchta, E. Césàro, H. Curjel and T. Gosset.